LOCAL EQUIVALENCES ON CHORDS INVERSION

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Abstract. The equivalence between the Neo-Riemannian functions and the inversion function I_n depends on the chosen triad chords. To solve this problem, twelve theorems on inversions I_n equivalent to functions PLR are presented, which substitute the geometric calculation in the circumference and the Tonnetz, by a formal manipulation of arithmetic calculation.

Keywords: Neo-Riemannian functions, PC-Set Theory, Musical Inversion, Chords Symmetries, Tonnetz.

1. Introduction

Neo-Riemannian functions PLR are ubiquitous in the analysis of symphonic music [Coh97], jazz, pop, and pop [Cap04]. In the Tonnetz it is known that the Neo-Riemannian function $C \xrightarrow{P} Cm$. If it breaks down into root A, third B and fifth C; the function is described $P(A, B, C) = (A, B-1, C)$. In numbers, the chord of C major is $A = 0$, $B = 4$, $C = 7$. Hence, $P(0, 4, 7) = (0, 3, 7)$ [ASAALP$^+$12].

We also know about the duality of the functions R and L with T_n, as observed in the commutative diagrams in [CFS]. As for the circumference of pitch classes $\mathbb{Z}_{12}$ and using the inversion of the group T/I it is true that $C \xrightarrow{I_7} Cm$ [For73]. Decomposing the inversion in letters, we have: $I_7(A, B, C) = (C, B-1, A)$. In numbers: $I_7(0, 4, 7) = (7, 3, 0)$. Therefore, exist the next equivalence $P(C) = I_7(C)$.

2. First Problem

Since the Neo-Riemannian functions PLR behave globally well for all major and minor chords; for example, $C \xrightarrow{L} Em$ and $D \xrightarrow{L} F\sharp m$. It would be expected that with the functions I_n the same would happen; that is, $C \xrightarrow{I_{11}} Em$ and $D \xrightarrow{I_{11}} F\sharp m$. However, this is not true. What happens when applying the inversion I_{11} is $D \xrightarrow{I_{11}} Dm$. Which means that I_{11} applied to the local case of D is a P Neo-Riemannian function and not a L function. So, $I_{11}(D) = P(D)$. It should be added that the same happens with other chords, you have to find the correct inversion I_n that is equivalent to the PLR function involved.

One way to find the equivalences is to graph or manipulate a paper triangle on a circle of integers modulo 12. Then, by reflection and rotation, we get the real equivalence of the function L on the chord of D major: $L(D) = I3(D)$. But, this process of manipulation and geometric verification for each chord implies calculation time, except for a table of all the equivalences. Then, the following question arises:

> What is the formula that allows us to immediately calculate the equivalence of the inversion I_n with the Neo-Riemmanian functions PLR?

Amiot [Ami17] investigated certain problems that arise in the equivalences between the function I_n and the PLR functions. He formulated, for example, why certain inversions are equivalent to the P function, but when applied to other sets

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of pitch classes (triad chords) they are equivalent to the R or L function [Ami17, p. 138]. These fascinating and incisive research questions greatly motivated the writing of this article.

3. CONSTRUCTIONS OF EQUIVALENCES $PLR \Longleftrightarrow I_n$

The first objective will be to compose the local equivalences table of the functions PLR and the function I_n of all major and minor chords.

Table 1
Equivalences

P	R	L
$C \xleftrightarrow{I_7} Cm$	$C \xleftrightarrow{I_4} Am$	$C \xleftrightarrow{I_{11}} Em$
$D\flat \xleftrightarrow{I_9} D\flat m$	$D\flat \xleftrightarrow{I_6} B\flat m$	$D\flat \xleftrightarrow{I_1} Fm$
$D \xleftrightarrow{I_{11}} Dm$	$D \xleftrightarrow{I_8} Bm$	$D \xleftrightarrow{I_3} F\sharp m$
$E\flat \xleftrightarrow{I_1} E\flat m$	$E\flat \xleftrightarrow{I_{10}} Cm$	$E\flat \xleftrightarrow{I_5} Gm$
$E \xleftrightarrow{I_3} Em$	$E \xleftrightarrow{I_0} C\sharp m$	$E \xleftrightarrow{I_7} G\sharp m$
$F \xleftrightarrow{I_5} Fm$	$F \xleftrightarrow{I_2} Dm$	$F \xleftrightarrow{I_9} Am$
$F\sharp \xleftrightarrow{I_7} F\sharp m$	$F\sharp \xleftrightarrow{I_4} D\sharp m$	$F\sharp \xleftrightarrow{I_{11}} A\sharp m$
$G \xleftrightarrow{I_9} Gm$	$G \xleftrightarrow{I_6} Em$	$G \xleftrightarrow{I_1} Bm$
$A\flat \xleftrightarrow{I_{11}} A\flat m$	$A\flat \xleftrightarrow{I_8} Fm$	$A\flat \xleftrightarrow{I_3} Cm$
$A \xleftrightarrow{I_1} Am$	$A \xleftrightarrow{I_{10}} F\sharp m$	$A \xleftrightarrow{I_5} C\sharp m$
$B\flat \xleftrightarrow{I_3} B\flat m$	$B\flat \xleftrightarrow{I_0} Gm$	$B\flat \xleftrightarrow{I_7} Dm$
$B \xleftrightarrow{I_5} Bm$	$B \xleftrightarrow{I_2} G\sharp m$	$B \xleftrightarrow{I_9} D\sharp m$

With the Table 1 is quickly find a pattern of equivalences between the functions PRL and I_n. For example, $+2$ is added or -2 subtracted to each inversion when the triad chords rise or falls chromatically by semitone. In this sense, $+4$ is added or -4 is subtracted from if an increase or decrease by tone. In that sense, the calculation for other intervals is derived immediately. Just find a specific inversion for a chord in a geometric way and add or subtract according to the desired interval.

On the other hand, in the general set there are two groups of chords that have the same functions I_n. For example, F and B have the same subscripts: I_5, I_2, I_9. So also $E\flat$ and A: I_1, I_10, I_5.

4. THEOREMS $PLR \Longleftrightarrow I_n$

Going forward in our reasoning and in the equivalences shown in Table 1, we will formulate the arithmetic calculation propositions of inversions equivalent to the functions PLR for each major chord (A, B, C) and minor chord (a, b, c).

It should be clarified that the arithmetic calculation for equivalence in inversions gives the result of the inversions only with the input elements of the function I_n, without knowing the image. For example:

$$\begin{aligned} I_3(4, 8, 11) =& (x, y, z) & \\ =& (-4, -8, -11) & \text{Invert} \\ =& (8, 4, 1) & \text{Reduce mod 12} \\ =& (8+3, 4+3, 1+3) & \text{Add subscript of } I \\ =& (11, 7, 4). & \end{aligned}$$

In this case, $(4, 8, 11)$ is the pre-image and $(11, 7, 4)$ is the image. Hence, it follows that the function I_3 transforms (A, B, C) into $(C, B - 1, A)$. Since the Neo-Riemannian functions PLR hold two notes in common while one of them changes, the image set has two notes from the pre-image set. In this case, due to the geometric inversion, it has been permuted in such a way that the root of the pre-image becomes the fifth of the image and the fifth of the pre-image becomes the root of the image. The third maintains its position but descends a semitone.

Theorem 4.1. For every major triad (A, B, C) it is true that the Neo-Riemannian function P is equal to the inversion function I_n, such that n is the sum of the root and the fifth modulo twelve of the chord in question. The formula is expressed:

$$P(A, B, C) = I_{n\equiv A+C \bmod 12}(A, B, C) \tag{1}$$

Proof. An inversion n performs a sign change, therefore:

$$I_n(A, B, C) \mapsto (-A, -B, -C).$$

The proposition states that $n = A + C$. Then it is added to each element of the triad chord:

$$[-A + (A + C), -B + (A + C), -C + (A + C)].$$

By Neo-Riemannian theory it is known that the non-deductible cancellation equation in the image of $P(A, B, C)$ is $B - 1$. Then, the equations modulo 12 are obtained:

$$\begin{aligned} -A + A + C &= \quad C \\ -B + A + C &= \quad B - 1 \\ -C + A + C &= \quad A \end{aligned}$$

Verifying in the equations the equality P and I_n for the case G = $(7, 11, 2)$:

$$\begin{aligned} -7 + 7 + 2 \pmod{12} &= 2 \pmod{12} \\ -11 + 7 + 2 \pmod{12} &= 10 \pmod{12} \\ -2 + 7 + 2 \pmod{12} &= 7 \pmod{12} \end{aligned}$$

In effect, $I_{7+2}(7, 11, 2)$ is Gm $(2, 10, 7)$.

We extend the formula for the general case $n + 1$:

$$P(A + 1, B + 1, C + 1) = I_{(n+1)\equiv(A+1)+(C+1) \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned} [-(A + 1)] + (A + 1) + (C + 1) &= (C + 1) \\ [-(B + 1)] + (A + 1) + (C + 1) &= [(B - 1) + 1] \\ [-(C + 1)] + (A + 1) + (C + 1) &= (A + 1) \end{aligned}$$

This verifies the formula and completes the proof. □

Theorem 4.2. For every minor triad (a, b, c) it is true that the Neo-Riemannian function P is equal to the inversion function I_n, such that n is the sum of the root and the fifth modulo twelve of the chord in question. The formula is expressed:

$$P(a, b, c) = I_{n\equiv a+c \bmod 12}(a, b, c) \tag{2}$$

Proof. Similar to theorem 3.1. From Neo-Riemannian theory, it is known that the non-deductible equation by cancellation in the image of $P(a,b,c)$ is $b+1$. Therefore, the formula $P(a+1,b+1,c+1) = I_{(n+1)\equiv(a+1)+(c+1) \bmod 12}$ is analogue, and moreover fulfill the equations:

$$\begin{aligned}
[-(a+1)] + (a+1) + (c+1) &= (c+1) \\
[-(b+1)] + (a+1) + (c+1) &= [(b+1)+1] \\
[-(c+1)] + (a+1) + (c+1) &= (a+1)
\end{aligned}$$

This verifies the proposition and completes the proof. ☐

Theorem 4.3. For every major triad (A,B,C) it is true that the Neo-Riemannian function R is equal to the inversion function I_n, such that n is the sum of the root and the third modulo twelve of the chord in question. The formula is expressed:

$$R(A,B,C) = I_{n\equiv A+B \bmod 12}(A,B,C) \tag{3}$$

Proof. Similar to the theorems of the function P. By Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $R(A,B,C)$ is $C+2$. Then, the following equations are deduced:

$$\begin{aligned}
-A+A+B &= B \\
-B+A+B &= A \\
-C+A+B &= C+2
\end{aligned}$$

Verifying in the equations for the case D $= (2,6,9)$:

$$\begin{aligned}
-2+2+6 &= 6 \\
-6+2+6 &= 2 \\
-9+2+6 &= 9+2
\end{aligned}$$

We extend the formula for the general case $n+1$:

$$R(A+1,B+1,C+1) = I_{(n+1)\equiv(A+1)+(B+1) \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned}
[-(A+1)] + (A+1) + (B+1) &= (B+1) \\
[-(B+1)] + (A+1) + (B+1) &= (A+1) \\
[-(C+1)] + (A+1) + (B+1) &= [(C+2)+1]
\end{aligned}$$

This verifies the formula and concludes the proof. ☐

Theorem 4.4. For every minor triad (a,b,c) it is true that the Neo-Riemannian function R is equal to the inversion function I_n, such that n is the sum of the third and the fifth modulo twelve of the chord in question. The formula is expressed:

$$R(a,b,c) = I_{n\equiv b+c \bmod 12}(a,b,c) \tag{4}$$

Proof. Similar to theorem 3.3. From Neo-Riemannian theory, it is known that the non-deductible equation by cancellation in the image of $R(a,b,c)$ is $a-2$. Then, the following equations are deduced:

$$\begin{aligned}
-a+b+c &= a-2 \\
-b+b+c &= c \\
-c+b+c &= b
\end{aligned}$$

Verifying in the equations for the case Dm $= (2,5,9)$:

$$\begin{aligned} -2+5+9 &= 2-2 \\ -5+5+9 &= 9 \\ -9+5+9 &= 5 \end{aligned}$$

We extend the formula for the general case $n+1$,

$$R(a+1,b+1,c+1) = I_{(n+1)\equiv(b+1)+(c+1) \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned} [-(a+1)]+(b+1)+(c+1) &= [(a-2)+1] \\ [-(b+1)]+(b+1)+(c+1) &= (c+1) \\ [-(c+1)]+(b+1)+(c+1) &= (b+1) \end{aligned}$$

This verifies the proposition and completes the proof. ☐

Theorem 4.5. For every major triad (A,B,C) it is true that the Neo-Riemannian function L is equal to the inversion function I_n, such that n is the sum of the third and the fifth modulo twelve of the chord in question. The formula is expressed:

$$L(A,B,C) = I_{n\equiv B+C \bmod 12}(A,B,C) \tag{5}$$

Proof. By Neo-Riemannian theory, it is known that the non-deductible equation by cancellation in the image of $L(A,B,C)$ is $A-1$. Then, the following equations are deduced:

$$\begin{aligned} -A+B+C &= A-1 \\ -B+B+C &= C \\ -C+B+C &= B \end{aligned}$$

Verifying in the equations for the case E $=(4,8,11)$

$$\begin{aligned} -4+8+11 &= 4-1 \\ -8+8+11 &= 11 \\ -11+8+11 &= 8 \end{aligned}$$

We extend the formula for the general case $n+1$,

$$L(A+1,B+1,C+1) = I_{(n+1)\equiv(B+1)+(C+1) \bmod 12}$$

. Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned} [-(A+1)]+(B+1)+(C+1) &= [(A-1)+1] \\ [-(B+1)]+(B+1)+(C+1) &= (C+1) \\ [-(C+1)]+(B+1)+(C+1) &= (B+1) \end{aligned}$$

This verifies the formula and completes the proof. ☐

Theorem 4.6. For every minor triad (a,b,c) it is true that the Neo-Riemannian function L is equal to the inversion function I_n, such that n is the sum of the root and the third modulo twelve of the chord in question. The formula is expressed:

$$L(a,b,c) = I_{n\equiv a+b \bmod 12}(a,b,c) \tag{6}$$

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Proof. From Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $L(a,b,c)$ is $c+1$. Then, the following equations are deduced:

$$\begin{aligned} -a+a+b &= \quad b \\ -b+a+b &= \quad a \\ -c+a+b &= \quad c+1 \end{aligned}$$

Verifying in the equations for the case Am $= (9,0,4)$

$$\begin{aligned} -9+9+0 &= \quad 0 \\ -0+9+0 &= \quad 9 \\ -4+9+0 &= \quad 4+1 \end{aligned}$$

We extend the formula for the general case $n+1$,

$$L(a+1,b+1,c+1) = I_{(n+1)\equiv(a+1)+(b+1) \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned} [-(a+1)]+(a+1)+(b+1) &= (b+1) \\ [-(b+1)]+(a+1)+(b+1) &= (a+1) \\ [-(c+1)]+(a+1)+(b+1) &= [(c+1)+1] \end{aligned}$$

This verifies the proposition and completes the proof. □

5. DEFINITIONS FOR $P^{\psi}L^{\psi}R^{\psi}$ PRIMES FUNCTIONS AND THE SECOND PROBLEM

Neo-Riemannian prime functions were studied as simple function compositions by Morris [Mor98]. They were als used by Capuzzo [Cap04] to analyze pop-rock music. We will construct the definition of each prime function to be denoted with the superscript ψ to differentiate it from ordinary PLR functions. The internal notation of the triad chords is maintained as in [ASAALP+12].

Definition 5.1.

$$P^{\psi}(A,B,C) = (A+1,B,C+1)$$

$$P^{\psi}(a,b,c) = (a-1,b,c-1)$$

$$R^{\psi}(A,B,C) = (C,A-2,B-2)$$

$$R^{\psi}(a,b,c) = (b+2,c+2,a)$$

$$L^{\psi}(A,B,C) = (B+1,C+1,A)$$

$$L^{\psi}(a,b,c) = (c,a-1,b-1)$$

The goal again for this part of article is to answer the question:

> What is the formula that allows us to immediately calculate the equivalence of inversions I_n with the Neo-Riemmanian prime functions $P^{\psi}L^{\psi}R^{\psi}$?

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6. THEOREMS $P^{\psi} L^{\psi} R^{\psi} \Longleftrightarrow I_n$

Theorem 6.1. For every major triad (A, B, C) it is true that the Neo-Riemannian function P^{ψ} is equal to the inversion function I_n, such that n is the sum of the root and fifth plus 1 modulo twelve of the chord in question. The formula is expressed:

$$P^{\psi}(A, B, C) = I_{n \equiv A+C+1 \bmod 12}(A, B, C) \tag{7}$$

Proof. Similar to the previous theorems. From Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $P^{\psi}(A, B, C)$ is B. Then, the following equations are deduced:

$$\begin{aligned} -A + A + C + 1 = &\quad C + 1 \\ -B + A + C + 1 = &\quad\quad B \\ -C + A + C + 1 = &\quad A + 1 \end{aligned}$$

Verifying in the equations for the case C $= (0, 4, 7)$

$$\begin{aligned} -0 + 0 + 7 + 1 = &\quad 7 + 1 \\ -4 + 0 + 7 + 1 = &\quad\quad 4 \\ -7 + 0 + 7 + 1 = &\quad 0 + 1 \end{aligned}$$

We extend the formula for the general case $n + 1$:

$$P^{\psi}(A + 1, B + 1, C + 1) = I_{(n+1) \equiv [(A+1)+(C+1)+1] \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned} [-(A + 1)] + [(A + 1) + (C + 1) + 1] &= [(C + 1) + 1] \\ [-(B + 1)] + [(A + 1) + (C + 1) + 1] &= (B + 1) \\ [-(C + 1)] + [(A + 1) + (C + 1) + 1] &= [(A + 1) + 1] \end{aligned}$$

This verifies the formula and completes the proof. ☐

Theorem 6.2. For every minor triad (a, b, c) it is true that the Neo-Riemannian function P^{ψ} is equal to the inversion function I_n, such that n is the sum of the root and fifth minus 1 modulo twelve of the chord in question. The formula is expressed:

$$P^{\psi}(a, b, c) = I_{n \equiv a+c-1 \bmod 12}(a, b, c) \tag{8}$$

Proof. By Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $P^{\psi}(a, b, c)$ is b. Then, the following equations are deduced :

$$\begin{aligned} -a + a + c - 1 = &\quad c - 1 \\ -b + a + c - 1 = &\quad\quad b \\ -c + a + c - 1 = &\quad a - 1 \end{aligned}$$

Verifying in the equations for the case Dm $= (2, 5, 9)$

$$\begin{aligned} -2 + 2 + 9 - 1 = &\quad 9 - 1 \\ -5 + 2 + 9 - 1 = &\quad\quad 5 \\ -9 + 2 + 9 - 1 = &\quad 2 - 1 \end{aligned}$$

We extend the formula for the general case $n + 1$:

$$P^{\psi}(a + 1, b + 1, c + 1) = I_{(n+1) \equiv [(a+1)+(c+1)-1] \bmod 12}$$

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Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned}
[-(a+1)]+[(a+1)+(c+1)-1] &= [(c+1)-1]\\
[-(b+1)]+[(a+1)+(c+1)-1] &= (b+1)\\
[-(c+1)]+[(a+1)+(c+1)-1] &= [(a+1)-1]
\end{aligned}$$

This verifies the proposition and completes the proof. ☐

Theorem 6.3. For every major triad (A, B, C) it is true that the Neo-Riemannian function R^{ψ} is equal to the inversion function I_n, such that n is the sum of the root and third minus 2 modulo twelve of the chord in question. The formula is expressed:

$$R^{\psi}(A,B,C) = I_{n\equiv A+B-2 \bmod 12}(A,B,C) \tag{9}$$

Proof. Similar to the previous theorems. By Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $R^{\psi}(A, B, C)$ is C. Then, the following equations are deduced:

$$\begin{aligned}
-A+A+C-2 &= B-2\\
-B+A+C-2 &= A-2\\
-C+A+C-2 &= C
\end{aligned}$$

Verifying in the equations for the case G $= (7, 11, 2)$

$$\begin{aligned}
-7+7+11-2 &= 11-2\\
-11+7+11-2 &= 7-2\\
-2+7+11-2 &= 2
\end{aligned}$$

We extend the formula for the general case $n + 1$:

$$R^{\psi}(A+1,B+1,C+1) = I_{(n+1)\equiv[(A+1)+(B+1)-2] \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned}
[-(A+1)]+[(A+1)+(B+1)-2] &= [(B+1)-2]\\
[-(B+1)]+[(A+1)+(B+1)-2] &= [(A+1)-2]\\
[-(C+1)]+[(A+1)+(B+1)-2] &= (C+1)
\end{aligned}$$

This verifies the proposition and completes the proof. ☐

Theorem 6.4. For every minor triad (a, b, c) it is true that the Neo-Riemannian function R^{ψ} is equal to the inversion function I_n, such that n is the sum of the third and fifth plus 2 modulo twelve of the chord in question. The formula is expressed:

$$R^{\psi}(a,b,c) = I_{n\equiv b+c+2 \bmod 12}(a,b,c) \tag{10}$$

Proof. By Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $R^{\psi}(a, b, c)$ is a. Then, the following equations are deduced:

$$\begin{aligned}
-a+b+c+2 &= a\\
-b+b+c+2 &= c+2\\
-c+b+c+2 &= b+2
\end{aligned}$$

Verifying in the equations for the case Em $= (4, 7, 11)$

$$\begin{aligned}
-4+7+11+2 &= 4\\
-7+7+11+2 &= 11+2\\
-11+7+11+2 &= 7+2
\end{aligned}$$

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We extend the formula for the general case $n+1$:

$$R^{\psi}(a+1,b+1,c+1)=I_{(n+1)\equiv[(b+1)+(c+1)+2] \bmod 12}$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned}
[-(a+1)]+[(b+1)+(c+1)+2] &= (a+1)\\
[-(b+1)]+[(b+1)+(c+1)+2] &= [(c+1)+2]\\
[-(c+1)]+[(b+1)+(c+1)+2] &= [(b+1)+2]
\end{aligned}$$

This verifies the proposition and completes the proof. ☐

Theorem 6.5. For every major triad (A,B,C) it is true that the Neo-Riemannian function L^{ψ} is equal to the inversion function I_n, such that n is the sum of the third and fifth plus 1 module twelve of the chord in question. The formula is expressed:

$$L^{\psi}(A,B,C)=I_{n\equiv B+C+1 \bmod 12}(A,B,C) \tag{11}$$

Proof. By Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $L^{\psi}(A,B,C)$ is A. Then, the following equations are deduced:

$$\begin{aligned}
-A+B+C+1 &= A\\
-B+B+C+1 &= C+1\\
-C+B+C+1 &= B+1
\end{aligned}$$

Verifying in the equations for the case A $=(9,1,4)$

$$\begin{aligned}
-9+1+4+1 &= 9\\
-1+1+4+1 &= 4+1\\
-4+1+4+1 &= 1+1
\end{aligned}$$

We extend the formula for the general case $n+1$:

$$L^{\psi}(A+1,B+1,C+1)=I_{(n+1)\equiv[(B+1)+(C+1)+1] \bmod 12}.$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned}
[-(A+1)]+[(B+1)+(C+1)+1] &= (A+1)\\
[-(B+1)]+[(B+1)+(C+1)+1] &= [(C+1)+1]\\
[-(C+1)]+[(B+1)+(C+1)+1] &= [(B+1)+1]
\end{aligned}$$

This verifies the proposition and completes the proof. ☐

Theorem 6.6. For every minor triad (a,b,c) it is true that the Neo-Riemannian function L^{ψ} is equal to the inversion function I_n, such that n is the sum of the root and the third minus 1 module twelve from the chord in question. The formula is expressed:

$$L^{\psi}(a,b,c)=I_{n\equiv a+b-1 \bmod 12}(a,b,c) \tag{12}$$

Proof. By Neo-Riemannian theory it is known that the non-deductible equation by cancellation in the image of $L^{\psi}(a,b,c)$ es c. Then, the following equations are deduced

$$\begin{aligned}
-a+a+b-1 &= b-1\\
-b+a+b-1 &= a-1\\
-c+a+b-1 &= c
\end{aligned}$$

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Verifying in the equations for the case Bm $(11, 2, 6)$

$$\begin{aligned} -11+11+2-1 &= \quad 2-1 \\ -2+11+2-1 &= \quad 11-1 \\ -6+11+2-1 &= \quad 6 \end{aligned}$$

We extend the formula for the general case $n+1$:

$$L^{\psi}(a+1, b+1, c+1) = I_{(n+1)\equiv[(a+1)+(b+1)-1] \bmod 12}.$$

Therefore, the following equations modulo 12 are obtained:

$$\begin{aligned} [-(a+1)] + [(a+1)+(b+1)-1] &= [(b+1)-1] \\ [-(b+1)] + [(a+1)+(b+1)-1] &= [(a+1)-1] \\ [-(c+1)] + [(a+1)+(b+1)-1] &= (c+1) \end{aligned}$$

This verifies the proposition and completes the proof. ☐

After the proof of theorems is finished, a comparative table of Neo-Riemannian functions and inversions is prepared.

Table 2

Equivalences

P^{ψ}	R^{ψ}	L^{ψ}
$C \xleftrightarrow{I_8} C\sharp m$	$C \xleftrightarrow{I_2} Gm$	$C \xleftrightarrow{I_0} Fm$
$D\flat \xleftrightarrow{I_{10}} Dm$	$D\flat \xleftrightarrow{I_4} A\flat m$	$D\flat \xleftrightarrow{I_2} G\flat m$
$D \xleftrightarrow{I_0} D\sharp m$	$D \xleftrightarrow{I_6} Am$	$D \xleftrightarrow{I_4} Gm$
$E\flat \xleftrightarrow{I_2} Em$	$E\flat \xleftrightarrow{I_8} B\flat m$	$E\flat \xleftrightarrow{I_6} A\flat m$
$E \xleftrightarrow{I_4} Fm$	$E \xleftrightarrow{I_{10}} Bm$	$E \xleftrightarrow{I_8} Am$
$F \xleftrightarrow{I_6} F\sharp m$	$F \xleftrightarrow{I_0} Cm$	$F \xleftrightarrow{I_{10}} B\flat m$
$F\sharp \xleftrightarrow{I_8} Gm$	$F\sharp \xleftrightarrow{I_2} C\sharp m$	$F\sharp \xleftrightarrow{I_0} Bm$
$G \xleftrightarrow{I_{10}} G\sharp m$	$G \xleftrightarrow{I_4} Dm$	$G \xleftrightarrow{I_2} Cm$
$A\flat \xleftrightarrow{I_0} Am$	$A\flat \xleftrightarrow{I_6} E\flat m$	$A\flat \xleftrightarrow{I_4} D\flat m$
$A \xleftrightarrow{I_2} A\sharp m$	$A \xleftrightarrow{I_8} Em$	$A \xleftrightarrow{I_6} Dm$
$B\flat \xleftrightarrow{I_4} Bm$	$B\flat \xleftrightarrow{I_{10}} Fm$	$B\flat \xleftrightarrow{I_8} E\flat m$
$B \xleftrightarrow{I_6} Cm$	$B \xleftrightarrow{I_0} F\sharp m$	$B \xleftrightarrow{I_{10}} Em$

As observed in Table 2, the pattern for the chromatic rise of the chords for the functions $P^{\psi}R^{\psi}L^{\psi}$ in functions I_n is equal to the one shown in table 1. Consequently, each semitone increases the inversion by 2; and each tone in 4.

7. CONCLUSIONS

We have found a solution to the equivalence calculation problem of the inversion function I_n for the simple functions PLR and $P^{\psi}R^{\psi}L^{\psi}$ from a local point of view with the aim of reducing the geometric calculation time. For this, twelve theorems were proved by induction. Likewise, the elaboration of two tables with all possible cases proves the theorems exhaustively.

Future work will focus on building and demonstrating musical triad chord inversion equivalencies for more general PLR compound Neo-Riemannian functions.

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